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The Gel'fand Space and the Shilov Boundary of the Banach algebra $\mathcal{A} \times_I \mathcal{I}$ with I-product

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ABSTRACT
Let $\mathcal{A}$ be a Commutative Banach algebra and $\mathcal{I}$ be a closed ideal in $\mathcal{A}$. We can define so called I-product on $\mathcal{A} \times \mathcal{I}$, which makes it a commutative Banach algebra with some suitable norm. It is denoted by $\mathcal{A} \times_I \mathcal{I}$. In this paper, the Gel'fand space and the Shilov boundary of $\mathcal{A} \times_I \mathcal{I}$ is characterised in terms of that of $\mathcal{A}$ and $\mathcal{I}$.

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KEYWORDS
I-product, Gel'fand space, Shilov boundary and semisimplicity.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras. Then the Gelfand space and the Shilov boundary of the cartesian product $\mathcal{A} \times \mathcal{B}$ are characterized in [3]. If $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$, then these two objects of direct-sum product $\mathcal{A} \times_d \mathcal{B}$ are characterized in [4]. Similarly, if $\mathcal{I}$ is a closed ideal of $\mathcal{A}$, then these objects of the convolution product $\mathcal{A} \times_c \mathcal{I}$ are characterized in [5]. Here we define another product on $\mathcal{A} \times \mathcal{I}$ motivated from the direct-sum product. Let $\mathcal{A}$ be an algebra and $\mathcal{I}$ be an ideal in $\mathcal{A}$. Then $\mathcal{A} \times_I \mathcal{I}$ is an algebra with pointwise linear operations and the *I- product* defined as

$$(a, x)(b, y) = (ab, ay + xb + xy) \quad ((a, x), (b, y) \in \mathcal{A} \times_I \mathcal{I}).$$

It is easy to verify that $\mathcal{A} \times_I \mathcal{I}$ is commutative (resp. unital) iff $\mathcal{A}$ is commutative (resp. unital). If $\mathcal{A}$ is a normed algebra, then $\mathcal{A} \times_I \mathcal{I}$ is a normed algebra with the norm $\|(a, x)\|_1 = \|a\| + \|x\|$ $((a, x) \in \mathcal{A} \times_I \mathcal{I})$. Further, if $\mathcal{A}$ is a Banach algebra and $\mathcal{I}$ is a closed ideal in $\mathcal{A}$, then $(\mathcal{A} \times_I \mathcal{I}, \|\cdot\|_1)$ is a Banach algebra too.

Remark 1.1. *Let $\|\cdot\|$ be a norm on an algebra $\mathcal{A}$ and $\mathcal{I}$ be an ideal of $\mathcal{A}$. Let $\|(a, x)\|_\infty = \max\{\|a\|, \|x\|\}$ $((a, x) \in \mathcal{A} \times_I \mathcal{I})$. Then $\|\cdot\|_\infty$ may not be an algebra norm on $\mathcal{A} \times_I \mathcal{I}$.*

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2. Basic Results

Throughout this paper, $\mathcal{A}$ is an algebra over the complex field $\mathbb{C}$ and $\mathcal{I}$ is an ideal in $\mathcal{A}$. Let $\mathcal{A}_{-1}$ denote the set of all quasi invertible elements of $\mathcal{A}$. If $\mathcal{A}$ is unital, then $\mathcal{A}^{-1}$ is the set of all invertible elements of $\mathcal{A}$. Further, $\sigma_{\mathcal{A}}(a)$ and $r_{\mathcal{A}}(a)$ denote respectively the spectrum and the spectral radius of a in $\mathcal{A}$. Then we have the following.

Proposition 2.1. *Let $(a, x) \in \mathcal{A} \times_I \mathcal{I}$. Then*

(1) $(a, x) \in (\mathcal{A} \times_I \mathcal{I})^{-1}$ *iff* $a + x, a \in \mathcal{A}^{-1}$;
(2) $(a, x) \in (\mathcal{A} \times_I \mathcal{I})_{-1}$ *iff* $a + x, a \in \mathcal{A}_{-1}$;
(3) $\sigma_{\mathcal{A}\times_I\mathcal{I}}((a, x)) = \sigma_{\mathcal{A}}(a + x) \cup \sigma_{\mathcal{A}}(a)$;
(4) $r_{\mathcal{A}\times_I\mathcal{I}}((a, x)) = \max\{r_{\mathcal{A}}(a + x), r_{\mathcal{A}}(a)\}$.

Proposition 2.2. *Let $\mathcal{A}$ be a normed algebra and $\mathcal{I}$ be a closed ideal in $\mathcal{A}$. Then $\mathcal{A} \times_I \mathcal{I}$ has a left approximate identity iff $\mathcal{A}$ has a left approximate identity. (Similar results are true for right, bounded left, bounded right approximate identity.)*

Proof. Let $\{(e_\alpha, x_\alpha)\}_{\alpha\in\Lambda}$ be a left approximate identity in $\mathcal{A} \times_I \mathcal{I}$ and $a \in \mathcal{A}$. Then

$$\|e_\alpha a - a\| \leq \|e_\alpha a - a\| + \|x_\alpha a\| = \|(e_\alpha, x_\alpha)(a, 0) - (a, 0)\|_1.$$

Thus $\{e_\alpha\}$ is a left approximate identity for $\mathcal{A}$.

Conversely, suppose that $\mathcal{A}$ has a left approximate identity $\{e_\alpha\}_{\alpha\in\Lambda}$. Then,

$$\|(e_\alpha, 0)(a, x) - (a, x)\|_1 = \|(e_\alpha a, e_\alpha x) - (a, x)\|_1 = \|(e_\alpha a - a)\| + \|(e_\alpha x - x)\|$$

for every $(a, x) \in \mathcal{A}\times_I\mathcal{I}$. Thus $\{(e_\alpha, 0)\}_{\alpha\in\Lambda}$ is a left approximate identity for $\mathcal{A}\times_I\mathcal{I}$. □

Definition 2.3. *Let $\mathcal{A}$ be an algebra and $\|\cdot\|$ be a norm on $\mathcal{A}$. Then*

(1) $\|\cdot\|$ *is a* uniform norm *if* $\|a^2\| = \|a\|^2$ $(a \in \mathcal{A})$.
(2) $\mathcal{A}$ *is a* uniform *algebra if it admits a complete uniform norm.*
(3) If $\mathcal{A}$ *is a* $*$*-algebra and* $\|a^*a\| = \|a\|^2 (a \in \mathcal{A})$*, then* $\|\cdot\|$ *is a* C^*-norm *on* $\mathcal{A}$.

Lemma 2.4. *Let $\mathcal{I}$ be an ideal in a normed algebra $(\mathcal{A}, \|\cdot\|)$. Define*

$$|(a, x)| := \max\{\|a + x\|, \|a\|\} \ \ ((a, x) \in \mathcal{A} \times_I \mathcal{I}).$$

Then

(1) $|\cdot|$ *is a norm on* $\mathcal{A} \times_I \mathcal{I}$;
(2) $|\cdot|$ *is a uniform norm on* $\mathcal{A} \times_I \mathcal{I}$ *iff* $\|\cdot\|$ *is a uniform norm on* $\mathcal{A}$;
(3) Let $\mathcal{A}$ *be a* $*$*-algebra and* $\mathcal{I}$ *be a* $*$*-ideal in* $\mathcal{A}$*. Then* $|\cdot|$ *is a* C^**-norm on* $\mathcal{A} \times_I \mathcal{I}$ *iff* $\|\cdot\|$ *is a* C^**-norm on* $\mathcal{A}$.

Proof. (1) It is easy.

(2) Let $|\cdot|$ be a uniform norm on $\mathcal{A} \times_I \mathcal{I}$. Then

$$\|a^2\| = |(a^2, 0)| = |(a, 0)^2| = |(a, 0)|^2 = \|a\|^2 \quad (a \in \mathcal{A}).$$

Thus $\|\cdot\|$ is a uniform norm on $\mathcal{A}$.

Conversely, suppose that $\|\cdot\|$ is a uniform norm on $\mathcal{A}$. Then

$$\begin{aligned}|(a,x)^2| &= |(a^2, ax+xa+x^2)| \;=\; \max\{\|a^2+ax+xa+x^2\|, \|a^2\|\}\\ &= \max\{\|(a+x)^2\|, \|a^2\|\} \;=\; \max\{\|a+x\|^2, \|a\|^2\}\\ &= \max\{\|(a+x)\|, \|a\|\}^2 \;=\; |(a,x)|^2\end{aligned}$$

for all $(a,x) \in \mathcal{A} \times_I \mathcal{I}$. Thus $|\cdot|$ is a uniform norm on $\mathcal{A} \times_I \mathcal{I}$.

(3) This can be proved as per statement (2).

□

Corollary 2.5. *Let $\mathcal{I}$ be a closed ideal in a Banach algebra $\mathcal{A}$. Then $\mathcal{A} \times_I \mathcal{I}$ is a uniform algebra if and only if $\mathcal{A}$ is a uniform algebra.*

Proof. Since $\mathcal{A} \cong \mathcal{A} \times \{0\}$ is a closed subalgebra of $\mathcal{A} \times_I \mathcal{I}$, $\mathcal{A}$ is a uniform algebra whenever $\mathcal{A} \times_I \mathcal{I}$ is a uniform algebra.

Conversely, let $\|\cdot\|$ be a complete uniform norm on $\mathcal{A}$. Then, by Lemma 2.4(2), $|\cdot|$ is a uniform norm on $\mathcal{A} \times_I \mathcal{I}$. Now we show that $|\cdot|$ is complete. Let $\{(a_n, x_n)\}$ be a Cauchy sequence in $(\mathcal{A} \times_I \mathcal{I}, |\cdot|)$. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned}\|x_n\| &\le \|a_n + x_n\| + \|a_n\|\\ &\le 2\max\{\|a_n + x_n\|, \|a_n\|\}\\ &= 2|(a_n, x_n)|.\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{I}, \|\cdot\|)$. Since $\|\cdot\|$ is a complete norm on $\mathcal{A}$ and $\mathcal{I}$ is closed in $\mathcal{A}$, the sequence $\{x_n\}$ converges to some $x \in \mathcal{I}$. By the similar argument, it follows that the sequence $\{a_n\}$ converges to some $a \in \mathcal{A}$. Hence the sequence $\{(a_n, x_n)\}$ converges to (a, x) in $(\mathcal{A} \times_I \mathcal{I}, |\cdot|)$. Thus $|\cdot|$ is a complete uniform norm on $\mathcal{A} \times_I \mathcal{I}$. □

3. Gel'fand Space and Shilov Boundary

Throughout this section, $\mathcal{A}$ is a commutative Banach algebra and $\mathcal{I}$ is a closed ideal in $\mathcal{A}$. In this section, we calculate the Gel'fand space $\Delta(\mathcal{A} \times_I \mathcal{I})$ and the Shilov boundary $\partial(\mathcal{A} \times_I \mathcal{I})$. Note that $\Delta(\mathcal{A} \times_I \mathcal{I})$ and $\partial(\mathcal{A} \times_I \mathcal{I})$ are similar to $\Delta(\mathcal{A} \times_d \mathcal{B})$ and $\partial(\mathcal{A} \times_d \mathcal{B})$ calculated in [4]. First we introduce some notations.

Notations: Let $\varphi \in \Delta(\mathcal{A})$. Define $\varphi^+, \varphi^\diamond : \mathcal{A} \times_I \mathcal{I} \longrightarrow \mathbb{C}$ as $\varphi^+((a,x)) := \varphi(a) + \varphi(x)$ and $\varphi^\diamond((a,x)) := \varphi(a)$ $((a,x) \in \mathcal{A} \times_I \mathcal{I})$. Let $F \subset \Delta(\mathcal{A})$. Define $F^+ := \{\varphi^+ : \varphi \in F\}$ and $F^\diamond := \{\varphi^\diamond : \varphi \in F\}$.

Theorem 3.1. $\Delta(\mathcal{A} \times_I \mathcal{I}) \cong \Delta^+(\mathcal{A}) \uplus \Delta^\diamond(\mathcal{A})$.

Proof. Let $\widetilde{\eta} \in \Delta(\mathcal{A} \times_I \mathcal{I})$. Define $\varphi(a) = \widetilde{\eta}((a,0))$ $(a \in \mathcal{A})$ and $\psi(x) = \widetilde{\eta}((0,x))$ $(x \in \mathcal{I})$. Then φ and ψ are multiplicative linear maps on $\mathcal{A}$ and $\mathcal{I}$, respectively such that

$\widetilde{\eta}((a,x)) = \varphi(a) + \psi(x)$ $((a,x) \in \mathcal{A} \times_I \mathcal{I})$. Now, for $(a,x),(b,y) \in \mathcal{A} \times_I \mathcal{I}$,

$$\begin{aligned} & \widetilde{\eta}[(a,x)(b,y)] = \widetilde{\eta}((a,x))\widetilde{\eta}((b,y)) \\ \Rightarrow\quad & \widetilde{\eta}((ab, ay+xb+xy)) = (\varphi(a)+\psi(x))(\varphi(b)+\psi(y)) \\ \Rightarrow\quad & \varphi(ab) + \psi(ay+xb+xy) = \varphi(a)\varphi(b) + \varphi(a)\psi(y) + \psi(x)\varphi(b) + \psi(x)\psi(y) \\ \Rightarrow\quad & \psi(ay) + \psi(xb) = \varphi(a)\psi(y) + \psi(x)\varphi(b). \qquad (3.1) \end{aligned}$$

Now, if $\psi \equiv 0$ on $\mathcal{I}$, then φ must be nonzero on $\mathcal{A}$. Therefore $\varphi \in \Delta(\mathcal{A})$. In this case, $\widetilde{\eta}((a,x)) = \varphi(a) = \varphi^\diamond((a,x))$ $((a,x) \in \mathcal{A} \times_I \mathcal{I})$. Thus $\widetilde{\eta} = \varphi^\diamond \in \Delta^\diamond(\mathcal{A})$. If $\psi \neq 0$ on $\mathcal{I}$, then there exists $y \in \mathcal{I}$ such that $\psi(y) \neq 0$. Now, taking $b = e$ and $a = x$ in Equation (3.1), we get $\psi(x) = \varphi(x)$ $(x \in \mathcal{I})$. Hence $\psi = \varphi$ on $\mathcal{I}$. Therefore, $\widetilde{\eta}((a,x)) = \varphi(a) + \psi(x) = \varphi(a) + \varphi(x) = \varphi^+((a,x))$ $((a,x) \in \mathcal{A} \times_I \mathcal{I})$. Thus, in this case, $\widetilde{\eta} = \varphi^+ \in \Delta^+(\mathcal{A})$. Thus $\Delta(\mathcal{A} \times_I \mathcal{I}) \subset \Delta^+(\mathcal{A}) \biguplus \Delta^\diamond(\mathcal{A})$. The reverse inclusion is trivial. Thus $\Delta(\mathcal{A} \times_I \mathcal{I})$ and $\Delta^+(\mathcal{A}) \biguplus \Delta^\diamond(\mathcal{A})$ are set theoretically equal. By arguments as in [3, Theorem 2.2], it can be shown that they are homeomorphic. □

Theorem 3.2. *[7, Corollary 3.3.4] Let X be a locally compact Hausdorff space, and let $\mathcal{A}$ be a subalgebra of $C_0(X)$ which strongly separates the points of X. Then a point $x \in X$ belongs to the Shilov boundary of $\mathcal{A}$ if and only if given any open neighbourhood U of x, there exist $f \in \mathcal{A}$ such that $\|f|_{X\setminus U}\|_\infty < \|f|_U\|_\infty$.*

Theorem 3.3. *Let $\mathcal{A}$ be a commutative Banach algebra and $\mathcal{I}$ be a closed ideal of $\mathcal{A}$. Then $\partial(\mathcal{A} \times_I \mathcal{I}) = \partial^+(\mathcal{A}) \biguplus \partial^\diamond(\mathcal{A})$.*

Proof. Let $\varphi_0 \in \partial\mathcal{A}$. Let $\widetilde{U}$ be a neighborhood of φ_0^+ in $\Delta(\mathcal{A} \times_I \mathcal{I})$. Then the set $U = \{\varphi \in \Delta(\mathcal{A}) : \varphi^+ \in \widetilde{U} \text{ or } \varphi^\diamond \in \widetilde{U}\}$ is a neighborhood of φ_0 in $\Delta(\mathcal{A})$. Therefore, by Theorem 3.2, there exists $a \in \mathcal{A}$ such that

$$\|\widehat{a}|_{\Delta(\mathcal{A})\setminus U}\|_\infty < \|\widehat{a}|_U\|_\infty.$$

If $\varphi^\diamond \in \Delta(\mathcal{A} \times_c \mathcal{I})\setminus\widetilde{U}$, then $(a,0)^\wedge(\varphi^\diamond) = \varphi(a)$. If $\varphi^+ \in \Delta(\mathcal{A} \times_c \mathcal{I})\setminus\widetilde{U}$, then $\varphi \in \Delta(\mathcal{A})\setminus U$ and $|(a,0)^\wedge(\varphi^+)| = |\varphi(a)|$. This gives

$$\|(a,0)^\wedge|_{\Delta(\mathcal{A}\times_c\mathcal{I})\setminus\widetilde{U}}\|_\infty = \|\widehat{a}|_{\Delta(\mathcal{A})\setminus U}\|_\infty.$$

Also $(a,0)^\wedge(\varphi^+) = \widehat{a}(\varphi) = (a,0)^\wedge(\varphi^\diamond)$ for every $\varphi \in \Delta(A)$. Hence

$$\|(a,0)^\wedge|_{\Delta(\mathcal{A}\times_c\mathcal{I})\setminus\widetilde{U}}\|_\infty = \|\widehat{a}|_{\Delta(\mathcal{A})\setminus U}\|_\infty < \|\widehat{a}|_U\|_\infty = \|(a,0)^\wedge|_{\widetilde{U}}\|_\infty.$$

Therefore, by Theorem 3.2, $\varphi_0^+ \in \partial(\mathcal{A} \times_I \mathcal{I})$. Thus $\partial^+(\mathcal{A}) \subset \partial(\mathcal{A} \times_I \mathcal{I})$. By Similar arguments it follows that $\partial^\diamond(\mathcal{A}) \subset \partial(\mathcal{A} \times_I \mathcal{I})$.

For the reverse inclusion, let $\widetilde{\varphi_0} \in \partial(\mathcal{A} \times_I \mathcal{I})$. Then $\widetilde{\varphi_0} = \varphi_0^+$ or $\widetilde{\varphi_0} = \varphi_0^\diamond$ for some $\varphi_0 \in \Delta(\mathcal{A})$.

case-I : $\widetilde{\varphi_0} = \varphi_0^+$ for some $\varphi_0 \in \Delta(\mathcal{A})$. Let U be a neighborhood of φ_0 in $\Delta(\mathcal{A})$. Then U^+ is a neighborhood of φ_0^+ in $\Delta(\mathcal{A} \times_I \mathcal{I})$. Since $\varphi_0^+ \in \partial(\mathcal{A} \times_I \mathcal{I})$, by Theorem 3.2, there exists $(a,x) \in \mathcal{A} \times_I \mathcal{I}$ such that

$$\|(a,x)^\wedge|_{\Delta(\mathcal{A}\times_I\mathcal{I})\setminus U^+}\|_\infty < \|(a,x)^\wedge|_{U^+}\|_\infty.$$

Hence

$$\|(a+x)^\wedge|_{\Delta(\mathcal{A})\setminus U}\|_\infty \leq \|(a,x)^\wedge|_{\Delta(\mathcal{A}\times_I\mathcal{I})\setminus U^+}\|_\infty < \|(a,x)^\wedge|_{U^+}\|_\infty = \|(a+x)^\wedge|_U\|_\infty.$$

Therefore $\varphi_0 \in \partial\mathcal{A}$.
case-II : $\widetilde{\varphi_0} = \varphi_0^\diamond$ for some $\varphi_0 \in \Delta(\mathcal{A})$. Let U be a neighborhood of φ_0 in $\Delta(\mathcal{A})$. Then $U^\diamond$ is a neighborhood of $\varphi_0^\diamond$ in $\Delta(\mathcal{A}\times_I\mathcal{I})$. Since $\varphi_0^\diamond \in \partial(\mathcal{A}\times_I\mathcal{I})$, by Theorem 3.2, there exists $(a,x) \in \mathcal{A}\times_I\mathcal{I}$ such that

$$\|(a,x)^\wedge|_{\Delta(\mathcal{A}\times_I\mathcal{I})\setminus U^\diamond}\|_\infty < \|(a,x)^\wedge|_{U^\diamond}\|_\infty.$$

Hence

$$\|a^\wedge|_{\Delta(\mathcal{A})\setminus U}\|_\infty \leq \|(a,x)^\wedge|_{\Delta(\mathcal{A}\times_I\mathcal{I})\setminus U^\diamond}\|_\infty < \|(a,x)^\wedge|_{U^\diamond}\|_\infty = \|a^\wedge|_U\|_\infty.$$

Therefore $\varphi_0 \in \partial\mathcal{A}$. Hence $\partial(\mathcal{A}\times_I\mathcal{I}) \subset \partial^+(\mathcal{A}) \uplus \partial^\diamond(\mathcal{A})$. □

Theorem 3.4. *Let $\mathcal{A}$ be a commutative Banach algebra and $\mathcal{I}$ be closed ideal in $\mathcal{A}$. Then $\mathcal{A}\times_c\mathcal{I}$ is semisimple if and only if $\mathcal{A}$ is semisimple.*

Proof. Let $\mathcal{A}\times_I\mathcal{I}$ be semisimple. Let $a \in \mathcal{A}$ such that $\varphi(a) = 0$ $(\varphi \in \Delta(\mathcal{A}))$. Then for any $\widetilde{\varphi} \in \Delta(\mathcal{A}\times_I\mathcal{I})$, $\widetilde{\varphi}((a,0)) = 0$. Since $\mathcal{A}\times_I\mathcal{I}$ is semisimple, $(a,0) = (0,0)$ gives $a = 0$. Thus $\mathcal{A}$ is semisimple.

Conversely, suppose that $\mathcal{A}$ is semisimple. Let $(a,x) \in \mathcal{A}\times_I\mathcal{I}$ be such that $\widetilde{\varphi}((a,x)) = 0$ $(\widetilde{\varphi} \in \Delta(\mathcal{A}\times_I\mathcal{I}))$. Let $\varphi \in \Delta(\mathcal{A})$. Then $\varphi^+, \varphi^\diamond \in \Delta(\mathcal{A}\times_I\mathcal{I})$. So that $\varphi^+((a,x)) = \varphi^\diamond((a,x)) = 0$. Implies that $\varphi(a) = \varphi(x) = 0$. Since $\varphi \in \Delta(\mathcal{A})$ is arbitrary and $\mathcal{A}$ is semisimple, we get $a = x = 0$. Hence $\mathcal{A}\times_I\mathcal{I}$ is semisimple. □

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